$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \quad , \text{by } [R_1 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -12 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A, \text{ by } \begin{bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 8R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 0 \end{bmatrix} A, \text{ by } R_3 \to R_3 - 4R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 4 & 0 \end{bmatrix} A, \text{ by } R_3 \to -R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ -1 & 5 & -2 \\ -1 & 4 & 0 \end{bmatrix} A, \text{ by } R_2 \to R_2 + R_3, \; R_1 \to R_1 - R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A, \text{ by } R_2 \to -\frac{1}{3} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A, \text{ by } R_1 \to R_1 - 2R_2$$

$$\therefore \quad I = A^{-1} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}.$$

3) Linear Independence of row and Column vectors

Definition: Let $\mathbb{F}$ ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) be a field then any element of n numbers $(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ is called an n-vector or simply a vector and an element $x_i \in F$, $i = 1, 2, \ldots n$ are called scalars.

Definition: Linearly Dependent set of vectors

Let $\{v_1, v_2, \ldots v_n\}$ be a set of n vectors then this form a linearly dependent set, if there exists n scalars $\alpha_1, \alpha_2, \ldots \alpha_n$, 'not all zero' such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0, \text{ where}$$

$0 = (0, 0, \ldots 0)$ is a n-zero vector.

Definition: Linearly Independent set of vectors

Let $\{v_1, v_2, \ldots v_n\}$ be a set of n vectors then this set is said to be linearly independent set if for all scalars, $\alpha_1, \alpha_2, \ldots \alpha_n \in F$

Whenever the relation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

then $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Definition: Linear Combination of Vectors

A vector v is said to be a linear combination of vectors $v_1, v_2, \ldots, v_n$ if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n.$$

Art. 1. If a set of vectors is linearly dependent, show that at least one member of the set is a linear combination of the remaining members.

Proof: Let $\{v_1, v_2, \ldots v_n\}$ is L.D. set.

$\Rightarrow \exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$$

let $\alpha_i \neq 0$.

then $\alpha_i v_i = -\alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} - \dots - \alpha_n v_n$

$\Rightarrow v_i = \frac{-\alpha_1 v_1}{\alpha_i} - \dots \frac{-\alpha_{i-1} v_{i-1}}{\alpha_i} - \frac{\alpha_{i+1} v_{i+1}}{\alpha_i} - \dots - \frac{\alpha_n v_n}{\alpha_i}$.

$\Rightarrow v_i$ is a linear Combination of $\{v_1, v_2, \dots v_{i-1}, v_{i+1}, \dots, v_n\}$.

Art. 2. If v is a linear Combination of the set of vectors $\{v_1, v_2, \dots v_n\}$ then the set $\{v, v_1, v_2, \dots, v_n\}$ is linearly dependent.

Solution: let $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars.

$\Rightarrow v - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_n v_n = 0$

$\Rightarrow \alpha \cdot v - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_n v_n = 0$, where $\alpha = 1$

$\Rightarrow$ Linear Combination of set of vectors $\{v, v_1, v_2, \dots, v_n\}$ is zero. But $\exists$ at least one scalar $\alpha = 1 \neq 0$.

$\Rightarrow \{v, v_1, v_2, \dots, v_n\}$ is L.D. set.

Art 3: Prove that every super set of linearly dependent set is linearly dependent.

Proof: let $\{v_1, v_2, \dots v_m\}$ be linearly dependent set and $\{v_1, v_2, \dots v_m, v_{m+1}, \dots v_n\}$ be super set of $\{v_1, v_2, \dots v_m\}$.

Now, $\{v_1, v_2, \dots v_m\}$ is linearly dependent set

$\Rightarrow \exists$ scalars $\alpha_1, \alpha_2, \dots \alpha_m$ not all zero such that

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + 0 \cdot v_{m+1} + \dots + 0 \cdot v_n = 0$.

where $\alpha_1, \alpha_2, \dots, \alpha_m, 0$ are scalars not all zero

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + 0 \cdot v_{m+1} + \dots + 0 \cdot v_n$ is a linear Combination of $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ such that $\exists$ atleast one scalar which is non-zero.

$\Rightarrow \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ is linearly dependent set.

Hence super set of a linearly dependent set is linearly dependent.

Art 4: Every subset of linearly independent set is linearly independent.

Proof: Let $\{v_1, \ldots v_n\}$ be linearly independent set and $\{v_1, v_2, \ldots v_m\}$ be its subset. Consider the scalars $\alpha_1, \alpha_2, \ldots \alpha_m$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + 0 \cdot v_{m+1} + \ldots + 0 \, v_n = 0$$

$\Rightarrow$ linear combination of $\{v_1, v_2, \ldots v_n\}$ is zero.

But $\{v_1, v_2, \ldots v_n\}$ is linearly independent set.

Therefore, $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$.

$\Rightarrow \{v_1, v_2, \ldots v_m\}$ is linearly independent set.

Hence subset of linearly independent set is linearly independent.

Definition: Row and Column vectors: Let $A = [a_{ij}]_{m \times n}$ be matrix of order $m \times n$ over F. i.e. $a_{ij} \in F$. Then the m rows $R_1, R_2, \ldots R_m$, where $R_i = (a_{i1}, a_{i2}, \ldots a_{in})$, $i = 1, 2, \ldots m$ is a n-vector i.e. $R_i \in F^n$ are called row vectors.

and $C_1, C_2, \ldots C_n$, where $C_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, $j = 1, 2, \ldots n$ is a m vector i.e. $C_j \in F^m$ are called column vectors.

(Linearly independent row (column) vectors): Row vectors $R_1, R_2, \ldots, R_m$ are said to be linearly independent if for any scalars $\alpha_1, \alpha_2, \ldots \alpha_m \in F$,

$$\alpha_1 R_1 + \ldots + \alpha_m R_m = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0.$$

Similarly, Column vectors $C_1, C_2, \ldots C_n$ are L.I. if for any scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$,